

# Math 206A Lecture 10 Notes

Daniel Raban

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## 1 $\mathcal{F}$ -Vectors of Polytopes

### 1.1 Types of polytopes

There are two types of convex polytopes in  $\mathbb{R}^d$ .

1. **simplicial polytopes** (all faces are simplices),
2. **simple polytopes** (degree of every vertex =  $d$ ,  $\dim(P) = d$ ).

There is a duality between these two types. Basically, a point on each face, and take the convex hull to get the dual polytope.  $P$  is simple iff  $P^*$  is simplicial.

**Definition 1.1.** Let  $P \subseteq \mathbb{R}^d$  be a convex polytope with  $\dim(P) = d$ . Let  $f_i(P)$  be the number of  $i$  dimensional faces of  $P$ . This is called the  **$\mathcal{F}$ -vector** of  $P$ .

**Proposition 1.1.**  $f_i(P) = f_{d-i-1}(P^*)$ .

Topologists like simplicial polytopes, but combinatorialists like simple polytopes. We will focus on simple polytopes, but the previous proposition tells us that this is really the same story.

### 1.2 Dehn-Sommerville equations

**Theorem 1.1** (Dehn-Sommerville equations<sup>1</sup>). *Let  $P \subseteq \mathbb{R}^d$  be simple. Then*

$$\sum_{i=k}^d (-1)^i \binom{i}{k} f_i = \sum_{i=d-k}^d (-1)^{d-i} \binom{i}{d-k} f_i.$$

for all  $0 \leq k \leq d$ .

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<sup>1</sup>The case  $d = 3$  was proved by Euler. The cases  $d = 4, 5$  were proved by Dehn, and  $d > 5$  was proved by Sommerville.

**Remark 1.1.** When  $k = 0$ , this becomes

$$\sum_{i=0}^d (-1)^i f_i = 1.$$

This is Euler's formula. When  $d = 3$ , we get  $f_0 - f_1 + f_2 = 2$ , where  $f_3 = 1$ .

**Example 1.1.** Let  $P$  be a simplex in  $\mathbb{R}^d$ . Then  $f_0 = d + 1$ , and  $f_i = \binom{d+i}{i+1}$ .

**Example 1.2.** Let  $Q \subseteq \mathbb{R}^d$  be a  $d$ -cube. Then  $f_0 = 2^d$ ,  $f_d = 1$ , and  $f_{d-1} = 2d$ . In general,  $f_i = \binom{d}{i} 2^{d-i}$ , because we have  $\binom{d}{i}$  ways to choose a face and  $2^{d-i}$  coordinates left. We get that

$$\sum_{i=0}^d f_i = 3^d,$$

which could be otherwise proven as an elementary exercise.

**Proposition 1.2.** Let  $\mathcal{F}(t) = \sum_{i=0}^d f_i t^i$ . Define  $\mathcal{G}(t) := \mathcal{F}(t - 1) = \sum_{i=0}^d g_i t^i$ . Then  $g_k = \sum_{i=1}^d (-1)^i \binom{i}{k} f_i$ .

*Proof.*

$$\mathcal{G}(t + 1) = \sum_{i=0}^d g_i (t + 1)^i = \sum_{i=0}^d g_i \sum_{k=0}^i \binom{i}{k} t^k = \sum_{k=0}^d t^k \left[ \sum_{i=k}^d g_i \binom{i}{k} \right] = \sum_{i=0}^d t^k f_k = \mathcal{F}(t). \quad \square$$

So the Dehn-Sommerville equations say that  $g_i = g_{d-i}$ .

**Example 1.3.** For a simplex,  $\mathcal{F}(t) = (1 + t)^{d+1}$ . Then  $\mathcal{G}(t) = t^{d+1}$ .

**Example 1.4.** For the  $d$ -cube,  $\mathcal{F}(t) = (2 + t)^d$ , and  $\mathcal{G}(t) = (1 + t)^d$ .

Let's prove the theorem.

*Proof.* Fix  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  a Morse function (a linear function that is nonconstant on edges of the polytope). For a vertex  $v$ , define the index  $\text{ind}_\varphi(v)$  to be the number of edges increasing by  $\varphi$ . Observe that  $0 \leq \text{ind}_\varphi(v) \leq d$ . Define  $h_i^{(\varphi)}$  to be the number of vertices  $v \in V(P)$  such that  $\text{ind}_\varphi(v) = i$ .

We claim that  $f_k = \sum_{i=k}^d \binom{i}{k} h_i^{(\varphi)}$ . Take any  $k$ -face  $Q$ , and let  $v$  be the minimum vertex with respect to  $\varphi$ . Let  $i = \text{ind}_\varphi(v)$ . The number of  $k$ -faces  $Q$  is  $\sum_{i=k}^d \binom{i}{k} h_i^{(\varphi)}$ , which is the number of ways to choose  $Q$  with minimum vertex  $v$  times the index.

Then  $\sum_{i=0}^d h_i^{(\varphi)} t^i = \mathcal{F}(t - 1)$ . So for all  $\varphi$ ,  $h_i^{(\varphi)} = g_i$ . If we replace  $\varphi$  with  $-\varphi$ , we get  $h_i^{(\varphi)} = h_{d-i}^{-\varphi}$ . The left hand side is  $g$ , and the right hand side is  $g_{d-i}$ .  $\square$